ON LANGUAGES WITH NON-HOMOGENEOUS STRINGS OF QUANTIFIERS*

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ABSTRACT
It is shown that every linear string of quantifiers can be replaced by a well-ordered sequence of quantifiers.

In Henkin [1] an attempt was made to generalize first-order languages. One of the directions was infinite languages, i.e. to add to the familiar logical operations (conjunction, negation, existential quantification, etc.) infinite operations. The definition of the truth value of infinite conjunctions (or disjunctions) was straightforward: $\bigwedge\{\phi_i : i \in I\}$ is made true by a given assignment, iff each ϕ_1 is made true by it. It is also intuitively clear what should be the meaning of an infinite homogeneous string of quantifiers (i.e. $(\exists x_1 \cdots \exists x_i \cdots)_{i \in I}\phi)$.

The meaning of a well-ordered string of quantifiers is, perhaps, also not difficult to grasp. Things become more complicated if one considers arbitrary linearly-ordered strings of quantifiers. To make an expression of the form $(\cdots Q_u x_u \cdots)_{u \in U}$ $\Phi[\cdots x_u \cdots]$, where U is linearly ordered and $Q_u \in \{\forall, \exists\}$, clear, one uses Skolem functions. Let $S = \{u \in U : Q_u = \forall\}$ and let $T = \{u \in U : Q_u = \exists\}$, then the expression above is equivalent to the existence of functions f_t, $t \in T$, such that f_t has the set of arguments $\{x_s : s \in S \; \& \; s < t\}$, where $<$ is the ordering of U, such that $\Phi(\cdots x_s \cdots f_t(\cdots x_s) \cdots)$ holds for all values of x_s, $s \in S$.

It is interesting to find out whether, making use of arbitrary linearly ordered strings of quantifiers, adds to the expressive powers of the language. The problem was stated by Keisler in [2]. In this paper an answer is given. We show that

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well-ordered strings of quantifiers are sufficient, in the sense that other quantifications can be reduced to them.

We will now state the problem and the result accurately.

Given a string of quantifiers, $\bar{Q} = (\cdots Q_u x_u \cdots)_{u \in U}$, where U is linearly ordered, define $u_1 E u_2$ to mean that $Q_{u_1} = Q_{u_2}$ and that, for every u between u_1 and u_2, $Q_{u_1} = Q_u$. It is obvious that E is an equivalentce relation, and that U is splitted into non-overlapping blocks which are the equivalence classes. Let $[u]$ be the equivalence class of u, then, if $u_1 < u_2$, $v_1 \in [u_1]$, $v_2 \in [u_2]$ and $[u_1] \neq [u_2]$, then $v_1 < v_2$. Thus $\{[u]: u \in U\}$, is linearly ordered in the obvious way. Its order type is denoted by '$t_p \bar{Q}$'.

We now will define a language $L(\lambda, \mu, \gamma)$, where λ and μ are infinite cardinals and γ is a set of linear order-types.

The formulas of $L(\lambda, \mu, \gamma)$ are obtained by the usual rules of formation of first-order formulas together with the following additional three clauses:

If $|I| < \lambda$ and $\{\Phi_i\}_{i \in I}$ is a set of formulas, then $\bigwedge_{i \in I} \Phi_i$ is a formula.

If Φ is a formula and $\bar{x}$ is a sequence, whose length is less than μ, of individual variables, then $\exists \bar{x} \Phi$ and $\forall \bar{x} \Phi$ are formulas.

If Φ is a formula and $\bar{Q} = (\cdots Q_u x_u \cdots)_{u \in U}$ is such that $|U| < \mu$ and $t_p \bar{Q} \in \gamma$ then $\bar{Q}\Phi$ is a formula.

The concept of a model is the same as the usual one, and if M is a model and τ is a mapping from the set of the individual variables into M, then the concept of satisfaction, '$M \models_\tau \Phi$' is defined as usual, with the following additional clauses:

$M \models_\tau \bigwedge_{i \in I} \Phi_i$ if $M \models_\tau \Phi_i$, for all $i \in I$. $M \models_\tau \exists \bar{x} \Phi$ if there is an assignment τ_1 which coincides with τ on every variable not occurring in $\bar{x}$, such that $M \models_{\tau_1} \Phi$, a similar clause being made for $\forall \bar{x} \Phi$.

$M \models_\tau \bar{Q}\Phi$ if, putting as before, $T = \{u : Q_u = \exists\}$, $S = \{u = Q_u = \forall\}$, there exists a family of functions, $\{f_t : t \in T\}$, such that f_t is a function of the arguments x_s, $s \in S$ and $s < t$, such that for every τ_1 which coincides with τ on every variable not occurring in $\bar{Q}$, and which satisfies $\tau_1(x_t) = f_t(\cdots \tau_1(x_s) \cdots)_{s < tas \in S}$, we have $M \Vdash_{\tau_1} \Phi$.

Two formulas Φ and Ψ are equivalent, if, for every model M and every τ, we have $M \models_\tau \Phi \Leftrightarrow M \models_\tau \Psi$.

THEOREM 1. *Let $\gamma_1 = \{\eta \in \gamma : \eta$ is a well-ordering$\}$. If there is a cardinal χ in γ_1, such that $|\eta| < \chi$, for all $\eta \in \gamma - \gamma_1$, and such that for all $\kappa < \chi$ we have $2^\kappa < \mathrm{Min}(\lambda, \mu)$, then every formula in $L(\lambda, \mu, \gamma)$ is equivalent to a formula in $L(\lambda, \mu, \gamma_1)$.*

The proof of Theorem 1 is rather messy and will not be given here. Its main idea can be gotten from the proof of Theorem 2.

THEOREM 2. *Put, as breore, $\gamma_1 = \{\eta \in \gamma: \eta$ is a well-ordering$\}$. If every η in $\gamma - \gamma_1$ is of the form α^*, where α^* is the inverse of α and α is a well-ordering such that $\alpha < \lambda$, then every formula in $L(\lambda, \mu, \gamma)$ is equivalent to a formula in $L(\lambda, \mu, \gamma_1)$.*

Proof. By induction on the formula. It is enough to show that if Φ is a formula of $L(\lambda, \mu, \gamma_1)$ and if $t_p\bar{Q} = \alpha^*$, where $\alpha < \lambda$, then $\bar{Q}\Phi$ is equivalent to a formula of $L(\lambda, \mu, \gamma_1)$. The proof is by induction on α. If $\alpha = 0$ the claim is trivial. If $\alpha = \delta + 1$ and the claim holds for δ, then it holds also for α since $\bar{Q}\Phi$ is either equivalent to $\exists \bar{x}\bar{Q}'\Phi$ or to $\forall \bar{x}\bar{Q}'\Phi$, where $t_p\bar{Q}' = \delta$. The only difficult case is where α is a limit ordinal. In that case if $\bar{Q} = (\cdots Q_u x_u \cdots)_{u \in U}$, then $U = \bigcup_{\beta<\alpha}[v_\beta]$, where $[v_\beta]$ are the disjoint blocks on which we have the induced ordering $\cdots < [v_\beta] < \cdots < [v_1] < [v_0]$, such that $Q_{u_1} = Q_{u_2}$, for every u_1, u_2 within the same block. For every $\delta < \alpha$ let $U_\delta = \bigcup_{\beta<\delta}[v_\beta]$ and let $\bar{Q}_\delta = (\cdots Q_u x_u \cdots)_{u \in U_\delta}$. Furthermore, let $S = \{u \in U: Q_u = \forall\}$, $T = \{u \in U: Q_u = \exists\}$. From now on s will range over S and t over T. By our induction hypothesis, each formula $\bar{Q}_\delta\Phi$ is equivalent to a formula of $L(\lambda, \mu, \gamma_1)$. Put

$$\Psi = (\cdots \forall x_s \cdots)_{s \in S}(\cdots \exists x_t \cdots)_{t \in T}[\bigwedge_{\delta<\alpha}\bar{Q}_\delta\Phi].$$

We claim that Ψ is equivalent to $\bar{Q}\Phi$. Since Ψ is equivalent to a formula of $L(\lambda, \mu, \gamma_1)$, the theorem follows from this claim.

$M \vDash_\tau \Psi$ iff there are functions g_t, $t \in T$, each g_t having the arguments $\cdots x_s \cdots$, $s \in S$ such that $\bigwedge_{\delta<\alpha}\bar{Q}_\delta\Phi$ holds with respect to every assignment τ_1 in which $\tau_1(x_t) = g_t(\cdots \tau_1(x_s) \cdots)$, and which coincides with τ on the other variables. $\bigwedge_{\delta<\alpha}\bar{Q}_{\delta\Phi}$ holds for τ_1 iff each $\bar{Q}_\delta\Phi$ does.

For a fixed δ, the variables x_u, $u \in U_\delta$, are quantified in $\bar{Q}_\delta\Phi$, hence the satisfaction of $\bar{Q}_\delta\Phi$ is not affected if the assignment is changed in an arbitrary way provided that the change is only for the x_u, $u \in U_\delta$.

Now $M \vDash_{\tau_1} \bar{Q}_\delta\Phi$ iff a family of suitable Skolem functions exists. For every $t \in T \cap U_\delta$ the corresponding Skolem function is a function of those x_s for which $s \in S \cap U_\delta$ and $s < t$; however, the functions would depend on the values which the assignment τ_1 assigns to the variables x_u where $u \in U - U_\delta$, for these are free in $\bar{Q}_\delta\Phi$. Hence, the satisfaction of Ψ is equivalent to the existence of the following families of functions: $\{g_t\}_{t \in T}$, mentioned before, and $\{h_{\delta,t}\}_{\delta<\alpha, t \in T\cap U}$, where each $h_{\delta,t}$ is a function of the x_u, where $u \in T - U_\delta$, and of the x_s, where $s < t$. These functions should take care of every $\bar{Q}_\delta \Phi$, meaning that, for every $\delta < \alpha$, Φ is satisfied with respect to every assignment of values to the variables x_u, $u \in U$, which is arrived by the following procedure (the values given to the other variables being those given by the original τ):

(I) the variables x_s, $s \in S$ are assigned arbitrary values;

(II) the value of each x_t, where $t \in T - U_\delta$ is to be determined by the values assigned in step (i) to the x_s, $s \in S$, according to the function g_t;

(III) the values of the x_s, $s \in S \cap U_\delta$, can now be changed in an arbitrary way;

(IV) every x_t, $t \in T \cap U_\delta$, is assigned a value which depends on the values of the x_u, $u \in U - U_\delta$, which were assigned in (i) and (ii) and the values of the x_s, $s \in S \cap U_\delta$ and $s < t$, assigned in (iii).

Now assume that $M \models_\tau \Psi$ and let the g_t and the $h_{\delta,t}$ satisfying the required conditions be given. We will show how to obtain from the g_t and the $h_{\delta,t}$ a family of functions $\{f_t\}_{t \in T}$, such that each f_t is a function of the x_s where $s < t$, and, for every assignment σ in which $\sigma(x_t) = f_t(\cdots \sigma(x_s) \cdots)_{s<t}$ and which coincides with τ on the other variables, we have $M \models_\sigma \Phi$. This will show that $M \models_\tau \Psi \rightarrow \bar{Q}\Phi$.

Consider all the indexed sets of members of M which are either of the form $\{a_s\}_{s<t}$, where $t \in T$, or of the form $\{a_s\}_{s \in S}$. Any member can appear more than once, and the indexed set should be vuisalized as ordered according to the ordering of the indices in U. Thus, each has an order-type which is coinitial with α^*. Say that two such indexed sets a and b are equivalent if, for some $u \in U$, a_s and b_s are both defined and equal for all $s < u$. The relation is easily seen to be an equivalence relation. Since each $\{a_s\}_{s<t}$ is equivalent to $\{a_s\}'_{s \in S}$, where $a'_s = a_s$ for $s < t$ and a'_s is arbitrary otherwise, it follows that every equivalence class has a member of the form $\{a_s\}_{s \in S}$. For each equivalence class E let $\phi(E)$ be a representative of the form $\{a_s\}_{s \in S}$.

Now let $\{a_s\}_{s<t}$ be given and define $f_t(\cdots a_s \cdots)$ as follows. Let $\{b_s\}_{s \in S}$ be $\phi(E)$ where E is the equivalence class of $\{a_s\}_{s<t}$. There are now two cases:

(i) For all $s < t$ we have $a_s = b_s$, then define $f_t(\cdots a_s \cdots)_{s<t}$ as $g_t(\cdots b_s \cdots)_{s \in S}$.

(ii) For some $s < t$ we have $a_s \neq b_s$. There is an ordinal β such that $a_s = b_s$ for all $s \in S - U_\beta$. Let δ be the minimal β having this property. For every $t_1 \in T - U_\delta$ let $a_{t_1} = g_t(\cdots b_s \cdots)_{s \in S}$. Define $f_t(\cdots a_s \cdots)$ to be $h_{\delta,t}(\cdots a_{t_1} \cdots a_s \cdots)_{t_1 \in T - U,_\delta s<t}$, that is, the value of $h_{\delta,t}$ for the case in which x_{t_1}, for $t_1 \in T - U_\delta$, is assigned the value

$g_{t_1}(\cdots b_s \cdots)_{s \in S'}$, and each x_s, where $s < t$, is assigned the value a_s.

Let $\{a_s\}_{s \in S}$ be given. Put $a_t = f_t(\cdots a_s \cdots)_{s<t}$. We have to show that Φ holds for the assignment σ such that $\sigma(x_u) = a_u$, $u \in U$. Let E be the equivalence class of $\{a_s\}_{s \in S}$ and let $\{b_s\}_{s \in S} = \phi(E)$. If, for all $s \in S$, $a_s = b_s$, them we have $a_t = g_t(\cdots a_s \cdots)_{s \in S}$ for all $t \in T$. Hence, in ths case, Φ holds because the assignment σ is obtained according to (I)–(IV) for the case $\delta = 0$. (Note that $U_0 = \emptyset$, hence for $\delta = 0$ there are no functions $h_{\delta,t}$.) In the general case there will be s that $a_s \neq b_s$. Let δ be the smallest ordinal β such that $a_s = b_s$ for all $s \in S - U_\beta$. The assignment σ is obtained according to (I)–(IV) as follows: First assign to each x_s, $s \in S$ the value b_s. In the second step assign to each x_t, where $t \in T - U_\delta$, the value which is determined by the function g_t, i.e. $g_t(\cdots b_s \cdots)_{s \in S}$. This value is exactly $f_t(\cdots a_s \cdots)_{s<t}$. This is so because, for $t \in T - U_\delta$, we have $a_s = b_s$ for all $s < t$, and since $\{a_s\}_{s \in S, s<t}$ and $\{a_s\}_{s \in S}$, are equivalent, clause (i) of the definition will yield: $f_t(\cdots a_s \cdots)_{s<t} = g_t(\cdots b_s \cdots)_{s \in S}$.

In the third step replace every b_s where $s \in S \cap U_\delta$ by a_s, and in the last step assign to x_t, where $t \in T \cap U_\delta$, the value determined by $h_{\delta,t}$, i.e.

$$h_{\delta,t}(\cdots a_u \cdots a_s \cdots)_{u \in T - U_\delta, s<t}.$$

This, as it is easy to see, is, by clause (ii) of the definition, equal to $f_t(\cdots a_s \cdots)_{s<t}$.

This concludes the proof in one direction.

The other direction, namely that $\bar{Q}\Phi$ implies Ψ is easy and is left to the reader.

References

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